An Introduction to Rotation Theory

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Abstract

This tutorial introduces one of the most fundamental dynamical systems by studying maps of the circle to itself. We are mainly going to investigate homeomorphisms of the circle. Homeomorphisms look easy at first sight, but this tutorial should convince you that this first impression is not quite correct. Indeed, we shall meet several surprising phenomena and discuss results, which demonstrate the complications arising already in 1-dimensional dynamics. Even in higher-dimensional problems it often turns out that we can reduce our system to a 1-dimensional setup, so that we are also trying to give major tools and results, which should be helpful even if you are not planning to stay in the 1-dimensional world. The theory associated with self-maps on the circle sometimes goes by the name "Rotation Theory" and as we shall see: This name is justified.

Contents

1 Preface

The tutorial is set up so that a beginning junior or senior undergraduate student from mathematics or a mathematics-related field with a basic course in analysis should be able to follow the main line of argument. Also, beginning graduate students with little background in dynamical systems beyond differential equations are hopefully able to benefit. If you are already beyond this point, you might be able to quickly read through and get a first idea about rotation theory and circle maps.

Naturally, some more advanced aspects of the subject are interesting, but difficult to incorporate. Therefore we summarized the part about measure-theoretic aspects of the subject in Section 4, which deals with ergodicity and recommend that you skip it if you have never studied measure theory or invest some extra time in learning some basic notions. Otherwise we only require results from "Classical Analysis" such as the Intermediate Value Theorem and some very basic facts about complex numbers.

Since the tutorial is inherently based on your access to the Internet, you need a working connection for some web-based animations of material presented here, these animations re-play automatically after a 3 second delay. There are also multiple exercises with hints and answers throughtout the tutorial and they form an essential part of it! You should do them!

None of the exercises require any long calculations. You should provide the key idea or a very short calculation, so there is no need to have scratch paper at hand.

Before I leave you to start, let me briefly make you aware of certain links you are going to find in the tutorial.

The links are an essential part of the tutorial, not much can be said on 10 pages, not much on 100 and even 1000 barely suffice to cover a small part of all of mathematics. So you are encouraged to explore via the links; if you are using a browser-plugin to read this pdf-file and open a link, you can just go back one page to return where you were in this file.

W signals a link to Wikipedia, the currently largest online encyclopdia. These links are meant to give you a way to directly look up some terminology.

M signals a link to Mathworld, one of the biggest mathematics resources available online. Again you can directly look up terminology here.

H signals a link to the biographies of mathematicians maintained by the Mathematics Department at the University of St. Andrews.

If you want to have many external pages available at the same time it is recommended that you download this pdf-file to your computer and do not use it inside a browser window.

2 Circle maps

The main object of study of this tutorial are dynamical systems on the circle. More precisely, we are going to investigate dynamics on the space $\mathbb{R}/\mathbb{Z}$. This space can conveniently be thought of as the unit interval $[0, 1]$ with the two endpoints 0 and 1 identified since this is precisely what the construction $\mathbb{R}/\mathbb{Z}$ [W||M] means: We take the real line and identify two elements if they differ by an integer. As basic a property we recall the following result.

Fact 2.1. *The circle is a compact metric space, where the metric $p(x, y)$ is defined by*

$$\min_{\gamma} \{\textit{length of } \gamma|\gamma : [0, 1] \to S^1, \gamma(0) = x, \gamma(1) = y, \gamma \textit{ continuous}\}$$

Observe that the metric here is, what it is intuitively supposed to be, namely the length of the shortest arc between two points in S^1. The definition seems too complicated if one sees it for the first time, but if you happen to know any Differential Geometry you will recognize it as quite natural (if you don't you might want to briefly look here).

It is customary to denote the circle by S^1. Beyond the construction via an interval, there is another important point of view, namely considering $C = \{z \in \mathbb{C}||z| = 1\}$ so that we identify a subset of the complex plane as our circle. We shall indicate explicitly if we use this point of view since we are mainly going to work with $\mathbb{R}/\mathbb{Z}$.

Furthermore it is an easy task to translate between these two spaces...

Exercise 1. *Find a homeomorphism from* $\mathbb{R}/\mathbb{Z}$ *to* $C = \{z \in \mathbb{C} | |z| = 1\}$ *(you don't need to verify all details).*

- *Give me a hint!*
- *Go to the answer.*

So far we have only described our underlying space and to have a dynamical system we need a map on this space as well. One of the easiest dynamical systems on S^1 is the map given by

$$R_a(x) = x + a \pmod 1 \tag{1}$$

for some $a \in \mathbb{R}$. For convenience we are going to omit the notation (mod 1) [W‖M] from now on and we are going to indicate clearly when a map is not defined on S^1. We easily see by looking at $e^{2\pi ix} \mapsto e^{2\pi i(x+a)} = e^{2\pi ia}e^{2\pi ix}$ that we are dealing with a rotation, when we consider R_a. We are interested in what happens to a starting value x under iteration. Consequently we consider the sequence

$$O(x) = \left\{R_a(x), R_a(R_a(x)), R_a(R_a(R_a(x))), \cdots, R_a^k(x), \cdots\right\} \tag{2}$$

where powers like "R_a^k" will always denote composition of maps. The set $O(x)$ is called the **forward orbit** of x. Obviously the **backward orbit** will consist of preimages.

The intuitive reason for trying to investigate $O(x)$ is that we might think of f as a transition map from one state of a physical system to another and x as a starting value. This idea might be familiar to you from the classical theory of differential equations. Just that in our case we have a discrete set of values instead of a solution curve, so that we could interpret our setup as working in discrete time.

As an example we have recorded the rotation map R_a "at work" for two different values of a:

- Click here to view animation 1
- Click here to view animation 2

Looking at the different maps we recognize that the behaviour seems very regular in the first case. Let us try to explain this; in this case the map actually has a rational parameter $a \in \mathbb{Q}$, so that we are talking about a rational rotation. The behaviour is not only regular as visually observed, but we have

Theorem 2.2. *If $a \in \mathbb{Q}$, then for any fixed x the forward orbit $O(x)$ is finite, which means that $R_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is periodic.*

That this result is correct can be visualized by keeping track of the iterates of a rational rotation, which is shown in the next animation.

- Click here to view animation 3

By now you might have thought already about a proof of this result and if you haven't then it is about time to do so!

Exercise 2. *Prove that a rational rotation is periodic.*
- *Give me a hint!*
- *Go to the answer.*

The second animation showed the iterates of a map R_a with an irrational parameter $a \in \mathbb{R} - \mathbb{Q}$. There is no directly visible pattern, but maybe we can see more if we also record all iterates on S^1.

- Click here to view animation 4

The observation is that the points are "uniformly distributed" on S^1 and we have to ask if it is possible to quantitatively describe this behaviour. Two natural questions are:

- Given some points $x, y \in S^1$. Does the orbit $O(x)$ contain a point "close" to y?
- Given two intervals in S^1 of different length, does the bigger interval contain "more points" of the orbit?

These questions are most easily answered by the fundamental theorem in the following section.

3 Weyl's Theorem

Theorem 3.1. *(Weyl [H]-Laplace [H]) Let a be irrational and consider the associated rotation $R_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $R_a(x) = x + a$. Let U be an interval in $\mathbb{R}/\mathbb{Z}$ of the form $U = [c, d]$ with $0 \leq c < d \leq 1$. Define*

$$T(x, n) = \# \left\{ 1 \leq k \leq n | R_a^k(x) \in U \right\} \tag{3}$$

(where "$\#S$" denotes the number of elements of a set S). Then we have

$$\lim_{n \to \infty} \frac{T(x, n)}{n} = c - d \tag{4}$$

Remark: The actual content of the result can be summarized by observing that with more and more iterations, the average number of points landing in an interval is equal to the length of the interval.

Proof. (Sketch) First we re-write the problem. Let $\chi_U(x)$ be the indicator function for U, i.e.

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Now we can express $T(x, n)$ in a different way

$$T(x, n) = \sum_{k=1}^{n} \chi_U(R_a^k(x))$$

Remember that our final goal is to prove $\lim_{n\to\infty} \frac{T(x,n)}{n} = c - d$. We have an equivalent form of this equation

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_U(R_a^k(x))}{n} = \int_0^1 \chi_U(x)dx \tag{5}$$

Exercise 3. *Convince youself that (5) is equivalent to our initial goal* $\lim_{n\to\infty} \frac{T(x,n)}{n} = c-d$. *Also think about a "physical interpretation" of each side in the last equation.*

- *Give me a hint!*
- *Go to the answer.*

Instead of proving the last equation for χ_U directly we are going to prove it first for continuous functions ϕ on the circle $\phi : \mathbb{R}/\mathbb{Z} \to C$, where $C = \{z \in \mathbb{C} | |z| = 1\}$. An easy approximation argument is going to give us the final result. So temporarily we aim at

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \phi(R_a^k(x))}{n} = \int_0^1 \phi(x)dx$$

Consider the set of functions $\left\{\phi_m(x) = e^{2\pi i m x} | |m| \leq n\right\}$. We need the following two facts, which are often discussed in a first course in Fourier Analysis [W][M]:

Fact 3.2. *The functions* $\left\{\phi_m(x) = e^{2\pi i m x} | |m| \leq n\right\}$ *form a basis in the space of trigonometric polynomials [W] on the circle.*

Fact 3.3. *Trigonometric polynomials are dense in the space of continuous functions on the circle.*

The first statement has a direct algebraic proof, whereas the density is a fundamental fact of Fourier [H] Analysis on the circle and is not immediate to prove. An elementary and also very enthusiastic account of the last fact is given in [Koerner89]. You could also view the density result as a direct consequence of this theorem.

In view of the previous facts we can use an approximation argument to establish the result for continuous functions if we can prove it for trigonometric polynomials of the form $\phi_m(x)$. Notice first that $\phi_m(R_a(x)) = e^{2\pi im(x+a)} = e^{2\pi ima}e^{2\pi im} = \mu_m\phi_m(x)$, where we have set $\mu_m = e^{2\pi ima}$. The previous relation can be used repeatedly to give $\phi_m(R_a^k(x)) = \mu_m^k\phi_m(x)$. Hence we calculate first for $m \neq 0$ and get

$$\sum_{k=1}^{n} \phi_m(R_a^k(x)) = \phi_m(x) \sum_{k=1}^{n} \mu_m^k = \phi_m(x)\frac{1-\mu_m^{n+1}}{1-\mu_m}$$

This immediately implies for the case $m \neq 0$ that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi_m(R_a^k(x)) = 0$$

If $m = 0$ we clearly have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi_m(R_a^k(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = \lim_{n\to\infty} \frac{n}{n} = 1$$

and therefore we have found the "time-average" in our desired equation for the case of trigonometric polynomials. The "space-average" is easily found by Euler's [H] formula $e^{ix} = \cos x + i \sin x$ and direct integration:

$$\int_0^1 e^{2\pi imx}dx = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$$

So we have shown that our equation holds for continuous functions, but we should remember that initially we dealt with $\chi_U(x)$, which is not continuous.

□

Exercise 4. *What is the basic approximation idea to get the result from continuous functions also for indicator functions [W] like $\chi_U(x)$? What properties/ideas could be used to make the approximation argument rigorous?*

- *Give me a hint!*
- *Go to the answer.*

Weyl's Theorem is also sometimes refered to as the "Equidistribution Theorem". Notice that the last proof illustrates the interplay between many concepts which are fundamental in classical analysis. It is often indispensable to be aware of techniques which might not appear to be directly related to problems in dynamics. Furthermore we can also answer a question, which we have posed initially about circle rotations.

Corollary 3.4. *If $R_a(x) = x + a$ is a circle rotation and a is irrational, then for any $x \in [0,1]$ the orbit of x is dense in $\mathbb{R}/\mathbb{Z}$.*

4 Ergodic Theory

This part of the tutorial puts the previous results in a bigger context and assumes a little bit of familiarity with some very basic notions of measure theory. You can skip this section if you want; no later section makes any use of the results stated here. It should be beneficial in any case to read on if you want a broader perspective.

First of all, since we have indeed found that the orbit of points is uniformly distributed over S^1, Weyl's Theorem can also be re-phrased in terms of probability theory. Recalling that we can look at the interval $[0, 1]$ with endpoints identified, another way of putting the last observation is that the probability distribution on $[0, 1]$ is uniform [W‖M].

Furthermore, Weyl's Theorem illustrates that a dynamical system can obey the property of having "space-averages" equal to "time-averages". That is, we can predict the system either by observing a single orbit or by taking an average over a spatial region. The theorem is indeed only one of the myriad of manifestations of a more general concept.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space [W‖M] and $f : X \longrightarrow X$ be measurable. Let us denote the orbit of an element $x \in X$ for simplicity by $x = x_0$, $f(x) = x_1$, $f^2(x) = x_2$, etc. and take a measurable real-valued function $\phi : X \longrightarrow \mathbb{R}$. Then we can also try to define a "time-average" $A(x_0)$ by

$$A_\phi(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(x_k) \tag{6}$$

where it is not a priori clear if this limit exists. If it does we have formally defined the time-average of ϕ over the forward orbit of x_0. We can also get a space-average of ϕ, which obviously should be defined by

$$S_\phi = \frac{1}{\mu(X)} \int_X \phi(x) d\mu(x) \tag{7}$$

The fundamental condition to impose on the map f is "ergodicity". To define this notion, recall that a set A is called **f-invariant** if $f(A) \subset A$. Now we call f **ergodic** if there does not exist any splitting of our measure space $X = X_1 \cup X_2$ into two f-invariant subsets X_1 and X_2 of strictly positive measure.

We like to think of f satisfying a certain minimality property with respect to the underlying measure space. Using this notion we have a remarkable theorem (see also the exercise following it for "deciphering"):

Theorem 4.1. *(Birkhoff [H] Ergodic Theorem) A measure preserving transformation $f : X \to X$ is ergodic if and only if for every integrable function ϕ we have $S_\phi = A_\phi(x)$ for all almost every $x \in X$.*

The proof of the Birkhoff Ergodic Theorem is beyond the scope of this tutorial, but the interested reader is referred to [K-H97].

Exercise 5. *Define the term* ***measure-preserving map*** f *between two measure spaces (don't cheat! - there is only one natural definition). Explain precisely, what is meant by "integrable function" and "almost every" in the previous theorem.*

- *Give me a hint!*
- *Go to the answer.*

It is easy to see that Weyl's theorem together with Birkhoff's Ergodic Theorem gives us that irrational rotations are ergodic. Indeed, it is immediate that a rotation as described above preserves the Lebesgue [H] measure induced on the circle S^1.

5 Rotation numbers

Now that the main dynamical properties of rational and irrational rotations are understood, it is natural to ask how these maps fit into the context of "general" maps on S^1. More precisely, we are now going to study homeomorphsims $f : S^1 \to S^1$. Suppose we are given a homeomorphism f and we can find another homeomorphism $h : S^1 \to S^1$ such that $R_a \circ h = f \circ h$, where R_a is a rotation given by $R_a(x) = x + a$. Then we call f **(topologically) conjugate** to R_a. In this case we can study the iteration of f using R_a since for $x \in S^1$

$$f^k(x) = \underbrace{h \circ R_a \circ h^{-1} \circ h \circ R_a \circ h^{-1} \circ \cdots h \circ R_a \circ h^{-1}}_{\text{k-times}} = h \circ R_a^k \circ h^{-1}$$

This calculation shows that we could generalize conjugacy [W‖M] outside of our setup of the circle. To remember how the compositions of maps work we can visualize the conjugate maps in a commutative diagram [W‖M]

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \Big\downarrow h & & \Big\downarrow h \\ S^1 & \xrightarrow{R_a} & S^1 \end{array}$$

Our goal is obviously to check, which homeomorphisms are indeed conjugate to rotations. A rotation $R_a(x) = x + a$ is completely characterized by the parameter a, so we should try to construct such a parameter for general homeomorphisms, which will be called "rotation number".

Notice that there is an obvious projection map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ given by $x \to x + \mathbb{Z}$. Based on this projection we can **lift** any homeomorphism $f : S^1 \to S^1$ to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$, which is required to satisfy $\pi \circ F = f \circ \pi$. [1] To illustrate geometrically how a lift works we can look at the following:

- Click here to view animation 5

We see that a lift essentially works by unfolding a map on S^1 to a map on $\mathbb{R}$. Recall that a homeomorphism $f : [0, 1] \to [0, 1]$ is called **orientation-preserving** if $x < y$ implies $f(x) < f(y)$, i.e. if "<" denotes the usual ordering then this is just monotonicity. We can extend this concept to S^1 by calling a circle homeomorphism **orientation-preserving** if it has a montone lift F such that $F(x + 1) = F(x) + 1$. The next result is going to enable us to define the rotation number of a homeomorphism $f : S^1 \to S^1$.

Theorem 5.1. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism and $F : \mathbb{R} \to \mathbb{R}$ be any lift. For $x \in \mathbb{R}$ define*

$$\tau(F) = \lim_{n\to\infty} \frac{1}{n}(F^n(x) - x) \tag{8}$$

Then this limit exists for all x and is independent of x. Furthermore $\tau(F)$ differs by an integer for two different lifts.

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[1] Notice that we do not have conjugacy here! Particularly topologists near you might become very angry if you claim any homeomorphism h between S^1 and $\mathbb{R}$.

Remark: We are only going to prove independence of x and the relation for different lifts below. The existence of the limit is an uninspiring exercise in analysis (see e.g. [K-H97]). We call $\pi(\tau(F)) = \tau(f)$ the **rotation number** of f.

Proof. (of Theorem 5.1) Since F is orientation-preserving we have $F(x+1) = F(x)+1$. This implies for any $x, y \in [0,1)$ that $|F(x) - F(y)| < 1$. Using this observation together with a direct triangle-inequality gives

$$\left|\frac{1}{n}|F^n(x) - x| - \frac{1}{n}|F^n(y) - y|\right| \leq \frac{1}{n}(|F^n(x) - F^n(y)| + |x - y|) \leq \frac{2}{n}$$

Taking the limit as $n \to \infty$ we see that the rotation numbers of x and y must be the same. That any rotation number just differs by an integer for different lifts is immediate if you have done the last exercise since $\tau(F+k) = \tau(F) + k$. □

The previous proof demonstrates that lifts are very convenient to work with. To convince yourself that we have defined the desired concept the following exercise is helpful:

Exercise 6. *Determine all possible lifts of a rotation $R_a(x) = x + a$ and calculate the rotation number. Observe also whether R_a is orientation-preserving or not.*

- *Give me a hint!*
- *Go to the answer.*

The rotation number is now well-defined and hopefully you are already worrying about the distinction between rational and irrational numbers. The guess to investigate these two cases seperately comes from our knowledge about rotations and it turns out to be a reasonable strategy.

6 Rational Rotation Numbers

Once you have really understood the examples we gave so far, it should be clear that periodicity is going to play a major role. The next theorem confirms this without any doubts!

Theorem 6.1. *If f is an orientation-preserving homeomorphism of S^1 then $\tau(f)$ is rational if and only if f has a periodic point.*

Proof. "⇒": Suppose that $\tau(f) = p/q \in \mathbb{Q}$. Using the definition of f we compute

$$\tau(f^m) = \lim_{n\to\infty} \frac{1}{n}(F^{mn}(x) - x) = m\tau(f) \pmod 1$$

Hence we conclude that $\tau(f^q) = 0$. Notice that we just used the familiar idea of showing periodicity for rational rotations in a different context. Therefore it is sufficient to show that if $\tau(f) = 0$, then f has a fixed point. We are going to proceed for this by contraposition, i.e. we assume f has no fixed point and want to get that $\tau(f) \neq 0$. Consider a lift F such that $F(0) \in [0, 1)$. If $F(x) - x \in \mathbb{Z}$ for some x we would have a fixed point for f, namely $\pi(x)$ (check this!). Therefore $F(x) - x \in \mathbb{R} - \mathbb{Z}$ and since $F - Id$ is continuous the Intermediate Value Theorem [W||M] gives us together with the construction $F(0) \in [0, 1)$ that

$$0 < F(x) - x < 1$$

Using continuity of $F - Id$ once more we conclude that on the compact set $[0, 1]$ the function $F - Id$ must achieve its maximum and minimum. So there exists a constant $\delta > 0$ such that

$$0 < \delta \leq F(x) - x \leq 1 - \delta < 1$$

Since $F - Id$ is periodic, the previous estimates hold for all $x \in \mathbb{R}$. So we simply pick a nice x, namely $x = F^k(0)$, and sum both sides of the previous inequality from $k = 0$ to $n - 1$ to obtain

$$n\delta \leq F^n(0) \leq n(1 - \delta)$$

On dividing by n and invoking independence of starting points of rotation numbers we get from the estimates

$$\delta \leq \frac{F^n(0)}{n} \leq 1 - \delta$$

that $\tau(f)$ cannot be equal to 0 as we wanted to show. The converse is explained in the next exercise. □

Exercise 7. *Prove the converse of the previous theorem, i.e. show that if f has a periodic point, then f must have a rational rotation number.*

- *Give me a hint!*
- *Go to the answer.*

It is not surprising that with a little bit more work using lifts we can also show that all periodic points must have the same period for an orientation-preserving homeomorphism with rational rotation number. This was expected from the usual rational rotations $R_{p/q}(x) = x + p/q$. This is the point, where the similarities end.

Let us look at the map given by

$$g(x) = x + \frac{1}{2} + \frac{1}{4\pi}\sin(2\pi x) \tag{9}$$

Several properties are easy to check and you should check a few to convince yourself:

- g has a rational rotation number; in particular $\tau(g) = \frac{1}{2}$
- The orbits consisting of $\left\{0, \frac{1}{2}\right\}$ $\left\{\frac{1}{4}, \frac{3}{4}\right\}$ show existence of periodic points.
- We can find a point $x \in [0, 1)$ such that x is nonperiodic!

Therefore the behaviour of homeomorphisms with rational rotation numbers needs special investigations with a focus on non-periodic orbits. The next two animations of g show what we should expect:

- Click here to view animation 6 - forward iteration
- Click here to view animation 7 - backward iteration

The visual observation is that our forward iterates approach the either 0 or 1/2 and the backward iterates approach either 1/4 or 3/4.

This behaviour occurs in far more general setups in dynamical systems at such a high rate that it deserves new terminology, namely if we are given a homeomorphism $f : S^1 \to S^1$ we call a point $x \in S^1$ **heteroclinic** to $y_1, y_2 \in S^1$ if

$$\lim_{n\to\infty} p(f^n(x), f^n(y_1)) = \lim_{n\to-\infty} p(f^n(x), f^n(y_2)) = 0$$

where p is the natural metric on S^1 introduced in the beginning of this tutorial (see Section 2). Roughly speaking, the orbit of x is asymptotically approaching the orbit of y_1 in forward time and y_2 in backward time. Alternatively we could simply call a point heteroclinic if it approaches a periodic orbit in forward time and another periodic orbit in backward time. This is precisely, what we have observed in our example for the map g.

Although we shall not need the next definition it is good to know some extra vocabulary in this situation, namely if we are only dealing with two points $x, y \in S^1$ and

$$\lim_{|n|\to\infty} p(f^n(x), f^n(y)) = 0$$

then x is **homoclinic** to y.

We are finally in a position to conclude the discussion for rational rotation numbers.

Theorem 6.2. *If $f : S^1 \to S^1$ is an orientation-preserving homeomorphism with a rotation number $\tau(f) = p/q \in \mathbb{Q}$, then a point x in a non-periodic orbit is heteroclinic under the map f^q to two points in a periodic orbit.*

We are not going to prove this result as this would take us too far outside the main line of argument. Nevertheless, the main idea of the proof is worth mentioning. We consider a lift of f and then view this map restricted to an interval $[x, x+1]$. Now this problem can be analyzed with tools from interval maps.

7 Limit Sets

We have seen that for rational rotation numbers we can gather a lot of information about the asymptotic behaviour of non-periodic points. The next goal is to determine the structure of such "asymptotic" or "limit" behaviour for irrational rotations. First, let us introduce another fundamental definition in dynamical systems. Let $f : S^1 \to S^1$ be a continuous map, then for $x \in S^1$ we define the **ω-limit set** of x as

$$\omega(x) = \left\{ y \in S^1 \middle| \text{ there exist } n_1 < n_2 < \cdots \text{ with } f^{n_k}(x) \to y \text{ as } k \to \infty \right\} \tag{10}$$

The ω-limit set contains all points, which come arbitrarily close to x, so we have indeed the formal notion of "limit" behaviour for points. Clearly, we can extend this definition to spaces beyond S^1 in a straightfoward way.

Exercise 8. *Fix any point $x \in S^1$. Find the ω-limit set $\omega(x)$ for the rotation $R_a(y) = y + a$, where $a \in \mathbb{R} - \mathbb{Q}$.*

- *Give me a hint!*
- *Go to the answer.*

As a minor result of the proof for the previous exercise we obtain that $\omega(x)$ is independent of $x \in S^1$ as long as we are dealing with our standard irrational rotation. This nice property persists for all orientation-preserving homeomorphisms with irrational rotation number, more precisely

Theorem 7.1. *Let f be an orientation-preserving homemorphism of S^1 with $\tau(f) \in \mathbb{R} - \mathbb{Q}$. Then $\omega(x) = \omega(y)$ for all $x, y \in S^1$.*

Remark: Regarding the following proof, try to extract which previously discussed key idea is used.

Proof. First we require an observation: If $I \subset S^1$ is an interval with endpoints $f^n(x)$ and $f^m(x)$ for some $n \neq m$ then every forward orbit $O(x) = \{f^n(x)\}_{n\in\mathbb{N}}$ must meet I. Before we prove this note that I cannot degenerate into a point since this would imply that we have a periodic point. This is impossible for irrational rotation numbers by our classification for rational rotation numbers ("rational" $\Leftrightarrow$ "$\exists$ a periodic point").

For our claim it obviously suffices to show that $S^1 = \bigcup_{k\in\mathbb{N}} f^{-k}(I)$ as this means the backward iterates of I cover the unit circle. Therefore every forward orbit of some x must meet I eventually. Set $I_k = f^{-k(n-m)}(I)$ and notice that for all $k \in \mathbb{N}$ the intervals I_{k-1} and I_k have at least one common endpoint. This implies that if $S^1 \neq \bigcup_{k\in\mathbb{N}} f^{-k}(I)$ then the sequence of endpoints must converge to some point x_0. Therefore

$$x_0 = \lim_{k\to\infty} f^{(-k+1)(n-m)} f^m(x) = f^{(n-m)} \lim_{k\to\infty} f^{-k(n-m)} f^m(x) = f^{(n-m)}(x_0)$$

which implies that x_0 is periodic, which is impossible as f has an irrational rotation number.

Now we are ready to attack the main problem so let $z \in \omega(x)$. Our goal is to show that $z \in \omega(y)$ since we can then simply reverse the roles of x and y to get that $\omega(x) = \omega(y)$. By definition of ω-limit sets there exists a sequence of natural numbers l_n such that $f^{l_n}(x) \to z$. By the previous observation we find another sequence $k_m \in \mathbb{N}$ such that $f^{k_m}(y) \in I_m = [f^{l_m}(x), f^{l_{m+1}}(x)]$. If we let $m \to \infty$ we see that $f^{k_m}(y) \to z$, which precisely means that $z \in \omega(y)$. $\square$

At least superficially some effort was made in proving a seemingly simple looking independence statement, but if you really understood the previous proof, then you can identify that the key part is to show that homeomorphisms with irrational rotation numbers have forward orbits meeting a certain interval eventually. For irrational rotations this would simply follow from Weyl's Theorem again.

8 Wandering Intervals

Our next goal is to examine the structure of $\omega = \omega(x)$ more closely. If you have not seen the next result before you are most likely very astonished:

Theorem 8.1. *Given any Cantor [H] set [W||M] $K \subset [0,1)$ and some irrational $\gamma \in (0,1)$, there is some homeomorphism f of S^1 such that $\omega(x) = K$ for all $x \in S^1$ and $\tau(f) = \gamma$.*

Remark: If you are at some point asked to give an explicit construction of ANY Cantor set in $[0,1)$ then all you need is a homeomorphism of the circle. We are only going to show the part regarding the Cantor set.

Proof. Our goal is to constructively obtain a map from the Cantor set K. First, enumerate all components of $S^1 - K$ in the same order as that of $\{n\gamma\}_{n\in\mathbb{Z}}$ on S^1 and call these I_n for $n \in \mathbb{N}$. Notice that for this procedure to work we need an order on S^1 and we might simply use the usual order of $[0,1)$ for this purpose. Next define the map $f : I_n \longrightarrow I_{n+1}$ by demanding that f is linear, onto and increasing. The map is illustrated in the following animation:

- Click here to view animation 8

Clearly we can extend f to S^1 continuously. Now let $x \in S^1 - K$ and observe that $O(x)$, the orbit of x under f, contains precisely one point in each component I_n so that $\omega(x) \subset K$. Since K is a Cantor set, it does not contain any isolated points, in particular every point in K is an accumulation point of a subsequence of I_n, which implies that $\omega(x) = K$. This concludes the main line of proof, the last step is your next exercise. □

Exercise 9. *In the last proof we have only shown that $\omega(x) = K$ for $x \in S^1 - K$, i.e. we have only considered points outside the actual Cantor set. Find a way to generalize this to S^1 to complete the proof of the theorem. Also justify, why $S^1 - K \neq \emptyset$.*

- *Give me a hint!*
- *Go to the answer.*

The construction of the homeomorphism in the previous theorem is an example of so-called "wandering" intervals. If you look at the next animation observe that the name captures the essence of the mapping we have defined, the interval "wanders around".

- Click here to view animation 8

Again we have encountered an important concept along the way. We call a nondegenerate open set $U \subset S^1$ **wandering** with respect to a continuous map $f : S^1 \to S^1$ if $f^n(U) \cap U = \emptyset$ for all $n > 0$ and $\bigcup_{x \in U} \omega(x)$ is not a single periodic orbit. The last condition says that under iteration the set U does not exhibit any periodic behaviour and the first condition means exactly that U "wanders around". We obviously call a set **non-wandering** if it is not wandering and also notice that it is easy to extend both definitions to other dynamical systems.

There is also a natural converse to our last result

Theorem 8.2. *If f is an orientation-preserving homemorphism of the circle with irrational rotation number $\tau(f)$, then $\omega(x) = S^1$ if there are no wandering intervals for f. If there are wandering intervals then $\omega(x)$ is a Cantor set.*

We are not going to prove this result since the ideas involved differ only slightly from our exposition so far. Having obtained quite a lot of information about the ω-limit set, we return to our initial question if we can indeed find a conjugation between an orientation-preserving homeomorphism f with irrational rotation number and an irrational rotation R_a. It turns out that the case of wandering intervals is especially troublesome.

Exercise 10. *Prove that if f has wandering intervals, then f is not conjugate to an irrational rotation $R_a(x) = x + a$.*

- *Give me a hint!*
- *Go to the answer.*

Though it turns out that this problem is the only one left and in all other cases we have succeeded with our classification

Theorem 8.3. *(Poincaré [H]) If an orientation-preserving homeomorphism f of S^1 has no wandering intervals and has an irrational rotation number $\tau(f)$ then f is conjugate to the irrational rotation $R_a(x) = x + a$ with $a = \tau(f)$.*

The proof uses a - unfortunately - very technical ordering argument, which we shall not present here. Instead we hope that during the last few paragraphs you have already raised the question

- Is there a situation or method to prevent wandering intervals?

There are esentially two answers to this questions, the first one is that one can simply try to eliminate certain bad points from S^1 and then try to conjugate. This procedure works in the usual manner of resolving problems inherent in a space - namely by considering a quotient construction [W||M]. The more satisfactory answer was given by Denjoy [H]. Before reading the result make sure you to recall or learn the definition of bounded variation [W||M] for a function.

Theorem 8.4. *(Denjoy) A C^1 diffeomorphism $f : S^1 \to S^1$ with irrational rotation number a and derivative of bounded variation is conjugate to the irrational rotation $R_a(x) = x + a$.*

Notice that we consider here the derivative of f as a function from S^1 to $\mathbb{R}$. Denjoy's theorem tells us that if we only consider C^1 maps then the dynamics is relatively "tame" in the sense that we have a model system with irrational rotations, which is easy to understand.

So you might think that the dynamical properties are not very interesting for C^1 diffeomorphisms, but then you should continue to the next part, which is going to change your opinion drastically.

9 Families of Diffeomorphisms

Clearly a single diffeomorphism is relatively nice, but what happens if we start studying families of diffeomorphisms? Instead of f, we are going to consider a familiy of maps $\{f_t\}$. This means we should also consider the rotation number as a map, i.e. we set

$$\tau : C^0 \to [0, 1) \tag{11}$$

so that τ becomes a map from continuous functions to $[0, 1)$. We cannot say very much about τ since as yet we have not even made C^0 into a metric space. It turns out to be useful to look at the uniform topology [W] by endowing C^0 with the metric given by $d(f, g) = \max_{x \in S^1} p(f(x), g(x))$.

Theorem 9.1. *τ is continuous in the uniform topology.*

Proof. Fix p/q and p'/q' in $\mathbb{Q}$ such that $p'/q' < \tau(f) < p/q$. Now construct a lift of f, say F, which gives the inequality

$$-1 < F^q(x) - x - p \leq 0$$

By continuity of $F - Id$ and the Intermediate Value Theorem [W][M] we see that actually $F^q(x) < x + p$ for all $x \in \mathbb{R}$. If not, then $F^q(x) = x + p$, which would mean that $\tau(f) = p/q$ contrary to our initial assumption. Furthermore $F^q - Id$ is periodic and continuous so it must attain its maximum. This allows us to find $\delta > 0$ such that $F^q(x) < x + p - \delta$, so if we perturb f in the uniform topology by δ then this perturbation $\bar{F}$ satisfies $\bar{F}^q(x) < x + p$ for all $x \in \mathbb{R}$. Similarly we can deal with the case for p'/q' to conclude that τ is a continuous map. □

Beyond the continuity we might want to extract more information about τ and as usual the standard example provides a hint how to do this. More specifically we fix a homeomorphism f and consider a new map $\rho(a) = \tau(R_a \circ f)$, where R_a is the usual rotation.

Exercise 11. *Show that if $f = R_b$ then $a \mapsto \tau(R_a \circ R_b) := \rho(a)$ is monotonically increasing.*

- *Give me a hint!*
- *Go to the answer.*

The situation is more complicated if we consider any homeomorphism for f. Using methods like lifts and the Intermediate Value Theorem [W||M] again, we can obtain the following:

Proposition 9.2. *The map $a \mapsto \rho(a) = \tau(R_a \circ f)$ is increasing and if $\rho(a)$ is irrational it is strictly increasing.*

Basically this means that whenever we "hit" an irrational rotation number for the familiy of maps $\{R_a \circ f\}$ and then increase a we are locally increasing. At rational numbers we might stay constant. This might seem paradoxical at first since we do not expect to have such a behaviour at rational numbers, but the intuition is deceiving here.

Proposition 9.3. *Suppose that $(R_a \circ f)^n \neq Id$ for all $a \in S^1$ and all $n \in \mathbb{N}$. Then for all $r/s \in \mathbb{Q}$ the preimage $\rho^{-1}(r/s)$ has non-empty interior. This implies also that the set $\{a|\rho(R_a \circ f) \in \mathbb{R} - \mathbb{Q}\}$ is nowhere dense [W||M] in S^1.*

Remark: The second part of the conclusion shows that there seems to be no way to get away without Cantor-type sets in this setup. Notice that we are still only in the setup of homeomorphism in dimension 1 on S^1!!

Proof. (Sketch) Denote by $K_{r/s}$ the set of values a such that $\rho(R_a \circ f) = r/s$. By continuity of the rotation number and its monotonicity we see that $K_{r/s}$ is non-empty. In particular we can write

$$K_{r/s} = \{a | (F(x) + a)^s(x) = x + r \text{ for some } x\}$$

where F denotes a lift of the map f. A similar construction as in the proof of continuity of the rotation number can now be employed to show that the following two sets are also non-empty and consist of single points

$$K^+_{r/s} = \left\{a \in K_{r/s} | (F(x) + a)^s(x) \geq x + r \text{ for all } x\right\}$$
$$K^-_{r/s} = \left\{a \in K_{r/s} | (F(x) + a)^s(x) \leq x + r \text{ for all } x\right\}$$

But since $(F(x)+a)^s(x)$ is not identically equal to $x+r$ for all a, we have $K^+_{r/s} \neq K^-_{r/s}$ and this means that $K_{r/s}$ must have non-empty interior, i.e. it does contain a non-empty interval. □

Hopefully you object immediately to the assumptions of this result! How do we know that we can find a homeomorphism f such that the condition $(R_a \circ f)^n \neq Id$ for all $a \in S^1$ and all $n \in \mathbb{N}$ is satisfied? There might not even be such a mapping f. Surprisingly, it turns out that there is even a diffeomorphism f of the circle, which satisfies the required hypothesis.

The following criterion will do the main work for us.

Lemma 9.4. *View S^1 as the unit circle C in the complex plane. Let $f : C \to C$ be an analytic [W‖M] diffeomorphism, which has an extension to an entire [W‖M] function in the complex plane $\mathbb{C}$, which is not an affine map. Then there is no integer n such that $f^n = Id$ on C.*

Proof. Suppose to the contrary that $f^n(x) = x$ on C. Then extend f to the complex plane, for convenience also denote it by f, and observe that the Identity Principle for holomorphic functions implies that $f^n(z) = z$ for all $z \in \mathbb{C}$. But this means that $f^{n-1} \circ f = Id$ so that f is an entire function with inverse f^{n-1}, which is also entire, i.e. we see that in this case f is biholomorphic [W]. From standard Complex Analysis it is well known that the only biholomorphic entire maps of the complex plane are affine maps, i.e. $f(z) = cz + d$ [W‖M] contradicting our initial assumptions. □

Again the idea of the proof is quite striking as it uses an unexpected trick to consider an extension of a map to the complex plane. You should be aware already that these surprises tend to occur quite frequently in dynamical systems!

The only task left is to come up with a nice analytic diffeomorphism which satisfies the assumption of our previous Lemma 9.4. It turns out that we can even produce a whole family of maps satisfying our assumptions. We define

$$f_b(x) = x + b\sin(2\pi x) \qquad \text{for } 0 < b < \frac{1}{2\pi} \tag{12}$$

It remains to check that we have chosen the correct map, so you should try the following exercise.

Exercise 12. *Consider the familiy of maps* $(R_a \circ f_b)(x) = x + b\sin(2\pi x) + a = f_{a,b}(x)$ *and show that* $f_{a,b}$ *defines a family of analytic diffeomorphisms of* S^1 *for* $b \in (0, \frac{1}{2\pi})$*, which have an entire extension to the complex plane.*

- *Give me a hint!*
- *Go to the answer.*

Hence we have constructed f so that Lemma 9.4 and Proposition 9.3 imply that for each fixed $b \in (0, \frac{1}{2\pi})$ the map

$$a \mapsto \rho(a) = \tau(R_a \circ f_b)$$

is monotone, continuous, locally constant at each value for which $\tau(R_a \circ f_b)$ is rational and non-constant at irrationals. Such a function is generally called a **devil's staircase**. Both of the links [W||M] show a picture of a devil's staircase, which is precisly using the construction we have indicated here!

Recall that you might have come across such a devil's staircase in connection with Cantor sets since another incarnation of a devil's staircase is constructed by defining a function to be constant and rational on the complement of a Cantor set in $[0, 1]$.

If you think this is not enough evidence that families of diffeomorphisms are complicated and give rise to interesting behaviour, notice that we have still one more trick left!

Namely look at the full familiy of maps

$$f_{a,b} = R_a \circ f_b$$

Using Proposition 9.3, we see that for a fixed irrational number ϕ the set $\{(a,b)|\tau(f_{a,b}) = \phi\}$ is the graph of a continuous function. If ϕ is rational it turns out that the set has non-empty interior and is bounded by continuous curves. The wedges between two such curves (or sometimes the whole system of curves) are called **Arnol'd tongues**. See the figure (1) from the book [D-V93] below for an illustration of these curves.

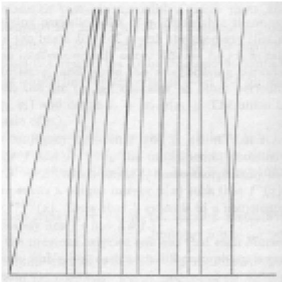

Figure 1: Illustration of Arnol'd tongues

The general phenomenon illustrated here is sometimes also refered to as **phase locking** or **mode locking** [W||M].

10 Further Steps and References...

The last figure for the Arnol'd tongues was taken from the excellent book of De Melo and Van Strien [D-V93]. If you are seriously considering learning more about dynamical systems in one dimension, this is a must-have as a reference text. It also covers another interesting aspect of circle homeomorphisms directly in the beginning of the book, namely that we can not only define the rotation number via lifts of maps and limits, but it turns out that it can be defined using combinatorics.

In particular to a homeomorphism f on S^1 we can associate a sequence of natural numbers $\{a_n\}$. The construction of this sequence is one of the involved technical aspects of the subject and requires a lot of book-keeping devices, i.e. indices, sub-indices, superscripts and a good chunk of the roman and the greek alphabet. This is the reason why you have not learned about this construction here. If you want to learn it, you should sit down with a lot of patience, scratch paper, and verify this construction on your own to actually understand it. The suprising result is that it we find in the end that

$$\tau(f) = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

Meaning that the rotation number can be defined in terms of a continued fraction expansion. The standard notation for this would be $\tau(f) = [0; a_1, a_2, a_3, \cdots]$. Obviously we have "extra" information encoded in the sequence a_n and we can use tools of symbolic dynamics [W] to show whether two homeomorphsims are equivalent, at least from the combinatorial point of view.

Combinatorics is emphasized in the book by Brucks and Bruin [B-B04], which starts from the fairly elementary, but goes quite far into modern directions of research like Farey [H] trees [W] and kneading theory. Another introductory text along similar lines (both books are actually appearing in the same series) is the text by Pollicott and Yuri [P-Y98], which stresses the measure-theoretic tools as it is also intended as an introduction to Ergodic Theory [W||M]. Several parts of this tutorial are based on these three books.

The main basis for the material we have seen so far is from the book by Katok and Hasselblatt [K-H97]. It makes a great reference text in many areas far beyond the material we have seen here, but is probably not the easiest entry point for the subject.

All four books mentioned so far [B-B04],[P-Y98],[K-H97],[D-V93] contain all necessary further references to get started with serious research in rotation theory and/or one-dimensional dynamical systems in general, therefore we are not going to reproduce these (mostly overlapping) lists here. If you want to persue the subject further, the following reading schedule might be very helpful:

1. Get a reference text such as [K-H97] or [D-V93]

2. Read the basic introduction to 1-dimensional dynamics in an advanced undergradute level course text such as [Devaney03] or a beginning graduate student text such as [B-S02].

3. Proceed to a more specialized text in one dimension. The author of this tutorial found [B-B04] and [D-V93] particulary helpful in this situation.

Random remarks about technology

This document was also created as an attempt to find out whether we can use a PDF-document to construct a tutorial, which takes advantage of many features available in a computer learning environment. As a side effect this also preserves the nice typesetting and fonts provided by LaTeX.

The only remaining problem with a PDF-document - at least in the humble opinion of the author of this document - seems to be that the integration of animations into PDF is not supported by LaTeX. You might object that this is not possible at all in PDF files! Surprisingly, this is not the case and some companies are offering to construct such files. But even after asking several experts for internet presentations, the author of this document could not find anybody knowing enough to teach him, how to do it...

So if you are able to get something like a gif-file into a PDF (or even better: you want to write a TeX-package to do this), please contact me: `ck274@cornell.edu`. Thank you!

References

[B-S02] M. Brin and G. Stuck, *Introduction to Dynamical Systems*, CUP, 2002
Go back to page: 38

[B-B04] K.M. Brucks and H. Bruin, *Topics from One-Dimensional Dynamics*, London Mathematical Society, Student Texts 62, CUP, 2004
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[D-V93] W. De Melo and S. Van Strien, *One-Dimensional Dynamics*, Springer, 1993
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[K-H97] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, CUP, 1997
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[Koerner89] T.W. Koerner, *Fourier Analysis*, CUP, 1989
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[P-Y98] M. Pollicott and M. Yuri, *Dynamical Systems and Ergodic Theory*, London Mathematical Society, Student Texts 40, CUP, 1998
Go back to page: 38

11 Hints

Hint 1.

Complex numbers have another nice representation except $z = x + iy$*...*

- *Go back to exercise 1*
- *To the answer of exercise 1*

Hint 2.

Write the parameter a as a rational number $a = \frac{p}{q}$ and observe that we are working "mod 1".

- *Go back to exercise 2*
- *To the answer of exercise 2*

Hint 3.

The first part is checking definitions and for the second part think about k *as time for the left-hand side and include a factor of* $\frac{1}{\int_0^1 dx}$ *on the right-hand side.*

- *Go back to exercise 3*
- *To the answer of exercise 3*

Hint 4.

Think about the indicator function of an interval and find a sequence of continuous functions converging to it.

Also notice that we are dealing with an equality of limits here (i.e. one limit of finite sums and one integral). Try to justify, why you can interchange the limit of your approximating sequence with the other limit opertions!

- *Go back to exercise 4*
- *To the answer of exercise 4*

Hint 5.

Measure-preserving should of course mean that the size of a measurable set remains unchanged. Integrability is a finiteness condition and certain small sets should be excluded if we want something to hold almost everywhere.

- *Go back to exercise 5*
- *To the answer of exercise 5*

Hint 6.

Think about what the projection map π does to integers...

- *Go back to exercise 6*
- *To the answer of exercise 6*

Hint 7.

We observe that if f has q-periodic point then $F^q(x) = x + p$ for some $p \in \mathbb{Z}$. Now think about the definition of rotation number for $mq = n$...

- *Go back to exercise 7*
- *To the answer of exercise 7*

Hint 8.

Recall Weyl's Theorem about the uniform distribution of orbits for irrational rotations. If you want a further hint, recall the following animation:

- *Click here to view animation 4*

- *Go back to exercise 8*
- *To the answer of exercise 8*

Hint 9.

This one is really quick if you paid attention to a previous result...

- *Go back to exercise 9*
- *To the answer of exercise 9*

Hint 10.

Think about what happens to the ω-limit set under conjugation...

- *Go back to exercise 10*
- *To the answer of exercise 10*

Hint 11.

Note that $R_a \circ R_b = R_{a+b}$.

- *Go back to exercise 11*
- *To the answer of exercise 11*

Hint 12.

Try to think how the function $f_{a,b}$ *looks like on* $[0,1]$*, e.g. for* $b = \frac{1}{4\pi}$ *and* $a = \frac{1}{2}$*. If you don't want to stretch your imagination you could use the nice tool here.*

For a formal proof you might want to look at the derivative of $f_{a,b}$ *viewed as a map on* $[0,1]$*.*

- *Go back to exercise 12*
- *To the answer of exercise 12*

12 Answers

Answer 1.

Define $h : \mathbb{R}/\mathbb{Z} \to C$ *by* $h(x) = e^{2\pi i x}$ *and observe that periodicity of the map makes this well-defined! Using the logarithm gives an inverse.*

- *Go back to exercise 1*

Answer 2.

Write $a = \frac{p}{q}$, *where* $p, q \in \mathbb{Z}$ *and* q *does not divide* p. *Considering* R_a^q *we see that*

$$R_a^q(x) = x + q \cdot \frac{p}{q} \ mod\ 1 = x + p \ mod\ 1 = x$$

so that R_a *is indeed periodic (we even have determined the number of elements in the orbit!).*

- *Go back to exercise 2*

Answer 3.

We try to prove that under iteration "time" and "space" averages agree, i.e.

$$\text{"time-average"} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_U(R_a^k(x))}{n} = \int_0^1 \chi_U(x)dx = \text{"space-average"}$$

- *Go back to exercise 3*

Answer 4.

The most elementary way is to approximate the indicator functions by two sequences of continuous functions from above and below as indicated in the following animation

- *Click here to view animation E1*

Now we can use the fact that our two sequences consist of uniformly continuous functions (since they are continuous on the circle, which is compact!) to interchange limits if necessary.

Alternatively, the Dominated Convergence Theorem [W‖M] could also do the job if you happen to know it.

- *Go back to exercise 4*

Answer 5.

Let $(X_1, \mathcal{C}_1, \mu_1)$ *and* $(X_2, \mathcal{C}_2, \mu_2)$ *be measure spaces then a measurable transformation* $f : X_1 \to X_2$ *is called* ***measure-preserving*** *provided that for every measurable set* $B \in C_2$ *we have* $\mu_1(f^{-1}B) = \mu_2(B)$*, where* $f^{-1}B$ *denotes the usual set-theoretic pre-image.*

In our theorem ϕ *is a map from* (X, μ) *to* $\mathbb{R}$ *and integrable simply means that* $\int_X \phi(x)\mu(x) < \infty$ *or otherwise said that* $\phi \in L^1(X, \mu)$*. Notice that it is a common convention to drop obvious terminology like the* σ*-algebra* $\mathcal{A}$ *and the Borel* σ*-algebra on* $\mathbb{R}$ *in the notation.*

Almost everywhere or "almost every" refers to the fact that the property holds except on a set of measure 0 *with respect to the measure* μ *in this case.*

- *Go back to exercise 5*

Answer 6.

If we define $R^*: \mathbb{R} \to \mathbb{R}$ *by* $R^*(x) = x + a + k$ *for some* $k \in \mathbb{Z}$, *then* $\pi \circ R_a = R^* \circ \pi$ *since* $\pi(k) = 0$. *By denoting with* π^{-1} *a map giving a preimage of an element in* S^1 *we see that any lift must satisfy that* R^* *differs from* R_a *by an integer. Now fix any lift* R^* *then we see that*

$$\lim_{n\to\infty} \frac{1}{n}((R^*)^n(x) - x) = \lim_{n\to\infty} \frac{1}{n}(x + na - x) = a \tag{13}$$

So the rotation number is precisely the generalization we wanted. Clearly, all lifts are just lines with slope 1, so they are monotone and satisfy $R^*(x+1) = R^*(x) + 1$, *which implies that rotations are orientation-preserving (as you would have guessed intuitively anyway!).*

- *Go back to exercise 6*

Answer 7.

We observe that if f has q-periodic point then $F^q(x) = x + p$ for some $p \in \mathbb{Z}$. For $m \in \mathbb{N}$ we just recall the definition of the quotient, which is relevant in the definition of the rotation number

$$\frac{F^{mq}(x) - x}{mq} = \frac{1}{mq}\sum_{i=0}^{m-1} F^q(F^{iq}(x)) - F^{iq}(x) = \frac{mp}{mq} = \frac{p}{q}$$

Hence the rotation number of F is given by $\tau(F) = p/q$.

- *Go back to exercise 7*

Answer 8.

An easy answer would be to invoke the result that orbits of irrational rotations are dense, which follows directly from Weyl's Theorem. If you want to use the definition of ω-limit sets as stated you could also write out the details like this:

We claim that $\omega(x) = S^1$ for all x. Take any $y_0 \in S^1$ and fix an interval I_1 centered at x with total width ϵ and $0 < \epsilon < 1$. Then Weyl's Theorem tells us that there exists some n_1 such that $f^{n_1}(y_0) \in I_1$. Call $f^{n_1}(y_0) = y_1$ and repeat the procedure for an interval I_2 centered at x of total width $\epsilon/2$. This gives a sequence of n_k such that

$$\lim_{k\to\infty} f^{n_k}(y_0) = x$$

This implies that $y_0 \in \omega(x)$ and since y_0 was arbitrary we conclude that $\omega(x) = S^1$.

- *Go back to exercise 8*

Answer 9.

Since $\tau(f)$ is irrational we have $\omega(x) = \omega(y)$ for all $x, y \in S^1$ (as we have shown in Theorem 7.1). Since K is a Cantor set, it is nowhere dense, so $S^1 - K \neq \emptyset$ and the theorem follows.

- *Go back to exercise 9*

Answer 10.

Notice that for a conjugation we would have a homeomorphism h from S^1 to itself such that $hfh^{-1} = R_a$ and the homeomorphism h would preserve the topological structure of the ω-limit set, which in the case of wandering intervals means we would map a Cantor set homeomorphically onto S^1, which is absurd.

- *Go back to exercise 10*

Answer 11.

Note that $R_a \circ R_b = R_{a+b}$. Monotonicity follows since we know that rotation numbers for R_a and R_{a+b} are given by a and $a+b$, so that we have an increasing function.

- *Go back to exercise 11*

Answer 12.

First, we look at the derivative of $f_{a,b}$ *viewed as a map on* $[0,1]$ *to see that* $f'_{a,b}(x) = 1 + \frac{2\pi}{b}\cos(2\pi x)$ *and observe that since* $b \in (0, \frac{1}{2\pi})$ *we have* $f'_{a,b}(x) > 0$ *for all* x *in* $[0,1]$*. This means* $f_{a,b}$ *is strictly monotonically increasing on* $[0,1]$*, so that we can invert.*

Also $f_{a,b}$ *is analytic since it is constructed from functions that are analytic by addition. The same reasoning holds for the question, whether we can extend* $f_{a,b}$ *to an entire function since the identity map and* $sin(x)$ *both extend to entire functions.*

Although you might think that this was a simple proof, the behaviour of the family of maps $f_b(x) = x + b\sin(2\pi x)$ *alone is not simple at all if we vary the parameter* b*! View the next animation to convince yourself.*

- *Click here to view animation E2*

- *Go back to exercise 12*